Gravity-Gradient Effects on the Motion of Two Rotating Cable-Connected Bodies

P. M. Bainum* and K. S. Evans†
Howard University, Washington, D.C.

Analytical and numerical methods are used to investigate the dynamics of a rotating two-body spacecraft system perturbed by first order gravity-gradient torques. The first order linear equations contain periodic coefficients with frequency at twice the system spin rate and the in-plane equations are forced at the same frequency. The amplitudes of all nonresonant steady-state motions are small. Conditions resulting in gravity-gradient induced resonances are examined analytically for special cases. The effect of damping on resonance is considered numerically.

Nomenclature

A	= coordinate system moving with the local vertical and located at the c.m. of the system; unit vectors $\hat{a}_i$, $i=1,2,3$.
$B(C)$	= coordinate system fixed in body 1(2) at its c.m. whose axes are principal axes of body 1(2); unit vectors $\hat{b}_i(\hat{c}_i)$
c.m.	= center of mass
$c_{B_i}(c_{C_i})$	= rotational spring constant for a restoring torque about the $B_i(C_i)$ axis
D	= coordinate system located at the system c.m. but with its first ordered axis along the cable line; unit vectors $\hat{d}_i$
$\mathfrak{F}$	= Rayleigh dissipation function
$I_{B_i}(I_{C_i})$	= moment of inertia of body 1(2) about the $B_i(C_i)$ axis
G	= universal gravitational constant
$(k_{B_i}k_{C_i})$	= rotational damping constant for a torque due to friction about the $B_i(C_i)$ axis
$k_1(k_2)$	= cable restoring (damping) constant
ℓ,ℓ_0,ℓ_e	= the instantaneous, unstretched, or equilibrium (respectively) cable length
M_e	= the mass of the Earth
m_i	= the mass of the end body, $i=1, 2$
R	= orbital radius
T	= total kinetic energy
t	= time
V	= total potential energy
$\alpha_1(\alpha_2)$	= coordinate measuring the variation of $\beta_3(\gamma_3)$ from its equilibrium value
$\beta_i(\gamma_i)$	= the ith angle in a 1-2-3 rotational sequence used to describe the orientation of body 1(2) with respect to the A system
δ	= dimensionless coordinate measuring the variation of ℓ from its equilibrium value $\delta=(\ell-\ell_e)/\ell_e$
θ_1	= angle in the orbit plane measuring the orientation of the cable line with respect to the A system
θ_2	= angle measuring the out-of-plane orientation of the cable with respect to the A system
$\dot{\theta}_n$	= the nominal spin rate of the system; also the equilibrium value of $\dot{\beta}_3$ and $\dot{\gamma}_3$
μ	= the reduced mass of the system $=m_1\ m_2/(m_1+m_2)$
$\rho_1(\rho_2)$	= the attachment length of body 1(2) (distance from the c.m. of body 1(2) to the point of cable attachment)
χ	= deviation of θ_1 from equilibrium value
Ω	= the orbital angular velocity of the system c.m.
$\omega_{B_i}(\omega_{C_i})$	= $B_i(C_i)$ component of the angular velocity of body 1(2)

Examples of dimensionless parameters:

$$\rho_i'=\rho_i/\ell_e,\quad i=1,2;\qquad I_{B_i}'=I_{B_i}/\mu\ell_e^2,\quad i=1,2,3$$

Other primed parameters:

$$c_{B_i}'=c_{B_i}/\mu\ell_e^2;\qquad k_{B_i}'=k_{B_i}/\mu\ell_e^2;\qquad i=1,2,3$$

$$k_i'=k_i/\mu;\qquad i=1,2$$

I. Introduction

A rotating cable-connected two-body spacecraft system has been proposed as a means of creating artificial gravity in space. Such a system, in its equilibrium rotational motion, would consist of a space station connected to a counterweight by a taut cable where the cable and its attachment points would all be coincident with the line connecting the mass centers of the two end bodies. As an alternate means of creating artificial gravity a single-part space station in a rim-like configuration could be rotated about its axis of symmetry. The first system may have certain weight and power system advantages over the second; to change the spin-rate of the system it is necessary to adjust the effective equilibrium length of the cable, whereas in the rim configuration an active power source is required.

In the area of rotating two-body spacecraft systems an earlier paper by Chobotov[1] included the effects of cable mass and elasticity for the case of point-mass end masses and two-dimensional motion. It was found that the gravity-gradient effects upon the small amplitude stability of the rotating system were very small and that the stability criteria could be expressed as a function of the cable natural frequencies, the angular velocities of the station and orbital motion, and the viscous damping parameters. Subsequently, Stabekis and Bainum[2] examined the motion and stability of a rotating space station-(massless) cable-counterweight configuration where the motion was restricted to the orbital plane. Although

Presented as Paper 75-157 at the AIAA 13th Aerospace Sciences Meeting, Pasadena, Calif., January 20-22, 1975; submitted January 17, 1975; revision received July 10, 1975. Research supported by NASA Grant: NGR 09-011-053; Special appreciation to V.N. Huff, MTE, NASA-Hq for many helpful suggestions.

Index categories: Spacecraft Attitude Dynamics and Control; Space Station Systems, Manned.

*Professor of Aerospace Engineering, Dept. of Mechanical Engineering. Associate Fellow AIAA.

†Graduate Student, Dept. of Mechanical Engineering;presently, Graduate Student, Dept. of Physics, Univ. of Chicago, Chicago, Ill. Associate Member AIAA.

the system remained stable in the absence of rotational damping (of end-body motions) it was found that this damping in addition to cable damping is necessary to achieve reasonable time constants for the nominal parameters considered. Nixon[3] deals with determining the dynamic equilibrium states in three dimensions for a completely undamped system with an arbitrary number of cables. He demonstrated using numerical integration that his linear system model predicted the motion most accurately when the angles do not deviate from equilibrium by more than one or two degrees. Analytic determination of the inplane and out-of-plane stability criteria was not attempted in this work. Anderson,[4] whose system had distributed end masses with lateral oscillations for three-dimensional motion, used an energy approach to analyze the motion of the system under the influence of disturbance torques. He found that the basic attitude response of the space station is that of an undamped second-order system and that coupled to this response are rigid body characteristics and cable lateral mode effects. In a recent paper Bainum and Evans[5] considered the three-dimensional extension to the problem considered in Ref. 2 in a torque-free environment. It was seen that for the linear system the out-of-(orbit) plane equations completely uncoupled form the in-plane equations. For the general case necessary conditions for in-plane stability were obtained, and necessary and sufficient conditions were obtained for out-of-plane motions.[5] In a related treatment Robe and Kane[6] considered the dynamics of a rotating system composed of two elastically connected, inertially identical, unsymmetrical rigid bodies, where the equations of motion were formulated in terms of the stiffness matrix. First-order gravitational torque effects were included and numerically shown to be small; in the subsequent stability analysis these terms were dropped with an appropriate warning of caution.

Of interest in this paper is a re-examination of the three dimensional motion of the rotating space station cable-counter-weight system of Ref. 5 to include first-order gravity-gradient torque effects. Allowances will be made for energy dissipation in both the cable system and end bodies as well as restoring torque effects.

II. Description of Mathematical Model

The mathematical model will be similar to that used in Ref. 5 except to include first-order gravitational torques. It is assumed that the system center of mass follows a circular orbit, that the cable is extensible but massless, and that the system at equilibrium has a nominal spin rate in the orbit plane about an axis passing through its center of mass.

Five different coordinate systems describe the motion. The fixed inertial reference (F) is located at the center of mass of the Earth, whereas, the A coördinate system is located at the center of mass of the system model with the A_1 axis along the local vertical, the A_2 axis in the direction of the velocity of the orbit and the A_3 axis normal to the orbit plane. The B system is fixed in the space station (body 1 as shown in Fig. 1) at its center of mass. The axes of the B system are assumed to be the principal axes of body 1 with the cable attached at a point on the B_1 axis. A one-two-three sequence of rotations, respectively, is assumed to orient the B system with respect to the A system. The C system fixed in a body 2 (the counterwieght) at its center of mass is defined the same way as the B system. Lastly, the D coordinate system is located at the center of mass of the model and is defined by two rotations with respect to the A system: an angle θ_1 in the orbit plane and then an angle θ_2 out of the plane. By these rotations the D_1 axis is parallel to the cable line.

The transformations from the A to the B systems, and from the A to the D systems are given in Eqs. (1) and (2) as:

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} c\beta_3 & s\beta_3 & 0 \\ -s\beta_3 & c\beta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta_2 & 0 & -s\beta_2 \\ 0 & 0 & 0 \\ s\beta_2 & 0 & c\beta_2 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\beta_1 & s\beta_1 \\ 0 & -s\beta_1 & c\beta_1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} c\theta_2 & 0 & s\theta_2 \\ 0 & 1 & 0 \\ -s\theta_2 & 0 & c\theta_2 \end{bmatrix} \begin{bmatrix} c\theta_1 & s\theta_1 & 0 \\ -s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \tag{2}$$

where c and s indicate cosine and sine functions, respectively. The transformation between the A and C systems is similar to Eq. (1) with β_i replaced by γ_i.

III. Development of the Equations of Motion

The following definitions of vectors are used through this section (see Fig. 2).

$r_{i/o}$ = geocentric position vector of the c.m. of body $i(i=1,2)$

$r_{i/A}$ = vector from the system c.m. to the c.m. of body $i(i=1,2)$

ℓ = vector from the attachment point of body 2 to the attachment point of body 1

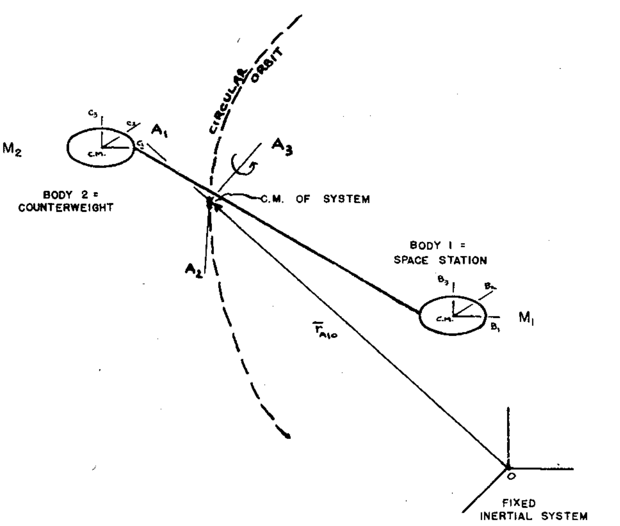

Fig. 1 System geometry.

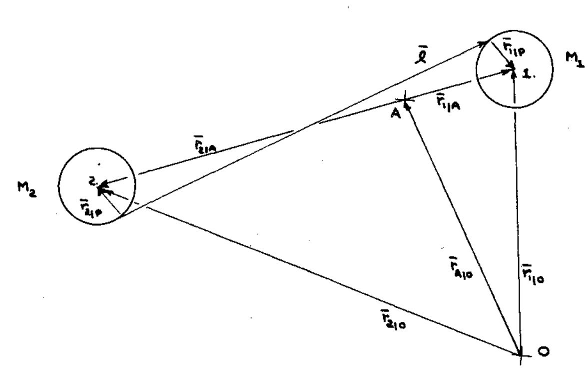

Fig. 2 Definition of vectors.

$r_{A/0}$ = geocentric position vector of the system c.m.
$r_{1/P}(r_{2/P})$ = vector from the attachment point of body 1 (2) to the c.m. of body 1(2)
$\omega_{B/A}$ = angular velocity of the B system with respect to the A system
$\omega_{A,B,C,D/F}$ = angular velocity of the A, B, C, or D systems with respect to the fixed inertial reference (F)
$(\dot{\ })_F$ = the time derivative of a vector with respect to the fixed, inertial reference
$(\dot{\ })_{A,B,C,D}$ = the time derivative of a vector with respect to the noninertial A, B, C, or D systems, respectively

A. Energy Expressions and the Rayleigh Dissipation Function

To use Lagrange's equations, it is necessary to express the total kinetic and potential energy of the system in terms of the variables which describe the system's motion. This can be performed by means of vector equations whose components are functions of the variables. The development of the system kinetic and potential energy is identical to that given in Ref. 5 (or, in more detail, Ref. 7) and will be repeated briefly here.

The equation for the translational kinetic energy is,

$$T_T = \tfrac{1}{2} m_1 \dot{r}_{1/0}^2 + \tfrac{1}{2} m_2 \dot{r}_{2/0}^2 \tag{3}$$

where $\dot{r}_{i/0}^2 = \dot{r}_{i/0} \cdot \dot{r}_{i/0}$ for $i = 1,2$. From the fact that the A system has a noninertial rotation, the inertial velocities of the centers of mass of the end bodies may be expressed

$$(\dot{r}_{i/0})_F = (\dot{r}_{i/A})_F + \omega_{A/F} \times r_{A/0}; \qquad i = 1, 2 \tag{4}$$

From the geometry of the system, (Fig. 2), we deduce

$$\ell + r_{1/P} - r_{1/A} + r_{2/A} - r_{2/P} = 0 \tag{5}$$

and from the fact that point A is the system c.m.,

$$m_1 r_{1/A} + m_2 r_{2/A} = 0 \tag{6}$$

Equations (5) and (6) can be combined in order to obtain expressions for the vectors $r_{1/A}$ and $r_{2/A}$. Then, after differentiation

$$\dot{r}_{1/A} = [m_2/(m_1 + m_2)]u \tag{7}$$

$$\dot{r}_{2/A} = -[m_1/(m_1 + m_2)]u \tag{8}$$

where

$$u = (\dot{\ell})_F + (\dot{r}_{1/P})_F - (\dot{r}_{2/P})_F \tag{9}$$

Then, after substituting Eqs. (7-9) into Eq. (4), the following is obtained

$$(\dot{r}_{1/0})_F = m_2 u/(m_1 + m_2) + \omega_{A/F} \times r_{A/0} \tag{10}$$

$$(\dot{r}_{2/0})_F = -m_1 u/(m_1 + m_2) + \omega_{A/F} \times r_{A/0} \tag{11}$$

With the aid of Eqs. (10) and (11), the translational kinetic energy, Eq. (3), may be written:

$$T_T = \tfrac{1}{2}\mu(u \cdot u) + \tfrac{1}{2}(m_1 + m_2)\,|\omega_{A/F} \times r_{A/0}|^2 \tag{12}$$

where the second term is constant for a circular orbit. The rotational kinetic energy has the form

$$T_R = \tfrac{1}{2} \sum_{i=1}^{3} (I_{B_i}\omega_{B_i}^2 + I_{C_i}\omega_{C_i}^2) \tag{13}$$

since the axes of the B and C systems are the principal axes of bodies 1 and 2, respectively.

The potential energy is, allowing for restoring forces in the cable and end bodies,

$$V = -\frac{GM_e(m_1 + m_2)}{r_{A/0}} + \frac{k_1(\ell - \ell_0)^2}{2} + \frac{1}{2}\sum_{i=1}^{3}(c_{B_i}\bar{\beta}_i^2 + c_{C_i}\bar{\gamma}_i^2) \tag{14}$$

where G is the gravitational constant, M_e is the mass of the Earth, and the bars indicate variational variables in the attitude angles. The interaction of on-board magnetics with the ambient magnetic field could provide rotational restoring torques derivable from this type of potential.[5] It is further assumed in connection with Eq. (14) that the restoring force in the cable is proportional to the linear extension from the unstretched length—i.e., the cable obeys Hooke's Law. (It should be noted that the first-order generalized forces associated with gravity-gradient torques to be considered in this paper will be derived directly from the torque expression and not from a corresponding potential energy function.)

Damping forces are assumed to act through a dissipation function of the form

$$\mathfrak{F} = \frac{1}{2}\left[\sum_{i=1}^{3}(k_{B_i}\dot{\bar{\beta}}_i^2 + k_{C_i}\dot{\bar{\gamma}}_i^2) + k_2\dot{\ell}^2\right] \tag{15}$$

where $\dot{\bar{\beta}}_i$ and $\dot{\bar{\gamma}}_i$ are the time rates of change of the variational variables corresponding to β_i and γ_i. This type of damping on the end bodies could be accomplished by employing three orthogonal magnetometers together with appropriate electronics.[5] It is also assumed that the cable will provide viscous damping proportional to the cable rate of extension.

B. Selection of Generalized Coordinates

The model used has two end masses. This indicates that for general unconstrained motion twelve variables are needed. If in addition, the variables θ_1, θ_2, and ℓ are added for the cable line, there will be fifteen variables in all. Eqs. (5) and (6) can be used to express $r_{1/A}$ and $r_{2/A}$ in terms of the other vectors. Since the x, y, z coordinates of each end mass are then eliminated, Eqs. (5) and (6) can be considered as six equations of constraint. Thus nine independent variables describe the motion of the model and are selected as follows: θ_1 and θ_2 for the orientation of the cable with respect to the A system; ℓ for the variable cable length; β_1, β_2, β_3 for the orientation of body 1 and γ_1, γ_2, γ_3, for the orientation of body 2.

As an example of how a quantity involved in the development can be expressed in terms of the generalized coordinates we consider that the angular velocity of a body is expressible in terms of the angles involved in the transformation matrix and the time rate of change of these angles. With the aid of transformation matrix Eq. (1), we can show

$$\omega_{B/A} = (\dot{\beta}_1 \cos\beta_3 \cos\beta_2 + \dot{\beta}_2 \sin\beta_3)\hat{b}_1 + (-\dot{\beta}_1 \sin\beta_3 \cos\beta_2 + \dot{\beta}_2 \cos\beta_3)\hat{b}_2 + (\dot{\beta}_1 \sin\beta_3 + \dot{\beta}_3)\hat{b}_3 \tag{16}$$

It is noted that $\dot{\beta}_1$ corresponds to a rotation about the A_1 axis, $\dot{\beta}_2$ corresponds to a rotation about the displaced A_2 axis, and $\dot{\beta}_3$ is associated with a rotation about the B_3 axis.

C. Lagrange's General Equation and the Procedure for Developing the Equations of Motion

With all expressions, kinetic, potential energies, etc., in terms of the nine generalized coordinates, Lagrange's equations,

$$\mathrm{d}/\mathrm{d}t(\partial T/\partial \dot{q}_j) - \partial(T - V)/\partial q_j = -\partial\mathfrak{F}/\partial\dot{q}_j + Q_{j_g} \tag{17}$$

yield the nine equations of motion for this space station-counterweight system. Expressions for the first-order generalized forces due to gravity-gradient torques, Q_{j_g}, will be presented in the next section.

The nine equations of motion were derived with the approximation that sin $(q_p) \cong q_p$ and cos $(q_p) \cong 1$ where q_p is any one of β_1, β_2, θ_2, γ_1 or γ_2 (angles out of the orbit plane). The approximation was made after the final differentation and all terms of degree higher than two in q_p were considered small. In addition it was assumed that the attachment arm vectors $r_{1/P}$, $r_{2/P}$ are in the direction of the unit vectors, $\hat{b}_1$ and $\hat{c}_1$, respectively; for the more general case where the cable attachment point is not located on the $B_1(C_1)$ principal axis, the equations would have to be appropriately modified.

In a stability analysis concerned with motion about an equilibrium state, variables are used which measure the deviation from the equilibrium motion. The following definitions were used in this respect

$$\theta_1 = \dot{\theta}_n t + \chi \qquad \delta = (\ell - \ell_e)/\ell_e$$

$$\beta_3 = \dot{\theta}_n t + \alpha_1 \qquad \gamma_3 = \dot{\theta}_n t + \alpha_2 \tag{18}$$

where $\dot{\theta}_n t$ is the equilibrium value of θ_1, β_3, and γ_3 for any time, t, and ℓ_e is the equilibrium cable length to be determined. χ, α_1, α_2, and δ are the new variational coordinates corresponding to θ_1, β_3, γ_3 and ℓ, the original variables. The original out-of-plane angles are zero at equilibrium and accordingly serve as variational coordinates. Thus, referring to Eqs. (14) and (15), $\bar{\beta}_3 = \alpha_1$, $\bar{\gamma}_3 = \alpha_2$ and $\bar{\beta}_i = \beta_i (i = 1, 2)$, $\bar{\gamma}_i = \gamma_i (i = 1, 2)$.

D. Development of First-Order Generalized Forces Due to Gravity-Gradient Torques

Robe[8] has used the following expression for the gravity-gradient torque about the mass center of the first end body of a tether-connected two body gravitationally stabilized system

$$T_{B_g} = 3\mathrm{GM_e}/2R^3 (\hat{a}_1 \times \bar{\bar{I}}_B \cdot \hat{a}_1) \tag{19}$$

where $\hat{a}_1$ is a unit vector in the direction of the system local vertical vector, and $\bar{\bar{I}}_B$ is the inertia dyad of body "1" with respect to the B coordinate system. Similarly, for the second end body,[8]

$$T_{C_g} = 3\mathrm{GM_e}/2R^3 (\hat{a}_1 \times \bar{\bar{I}}_C \cdot \hat{a}_1) \tag{20}$$

It should be noted that these expressions are first-order approximations[8] and would not be valid for the case of very long separation distances between the two end bodies.

Under the same assumptions given in Section II C, T_{B_g} and T_{C_g} can be written in terms of the nine independent coordinates of this system. If N_g is the total gravitational torque on the system, then it can be shown that

$$N_g = T_{B_g} + T_{C_g} = T_2 \hat{a}_2 + T_3 \hat{a}_3 \tag{21}$$

where expressions for T_2 and T_3 can be developed, assuming all second-degree and higher terms in the out-of-plane angular coordinates are small, as follows

$$\begin{aligned} T_2 = (3\mathrm{GM_e}/2R^3)[&-(I_{B_1}-I_{B_2})\beta_1 \sin 2(\dot{\theta}_n t + \alpha_1) \\ &+ (I_{B_1} - 2I_{B_3} + I_{B_2})\beta_2 + (I_{B_1} - I_{B_2})\beta_2 \cos 2(\dot{\theta}_n t + \alpha_1) \\ &- (I_{C_1} - I_{C_2})\gamma_1 \sin 2(\dot{\theta}_n t + \alpha_2) + (I_{C_1} - 2I_{C_3} + I_{C_2})\gamma_2 \\ &+ (I_{C_1} - I_{C_2})\gamma_2 \cos 2(\dot{\theta}_n t + \alpha_2)] \end{aligned} \tag{22}$$

and

$$\begin{aligned} T_3 = (3\mathrm{GM_e}/2R^3)[&(I_{B_1} - I_{B_2})\sin 2(\dot{\theta}_n t + \alpha_1) \\ &+ (I_{C_1} - I_{C_2})\sin 2(\dot{\theta}_n t + \alpha_2)] \end{aligned} \tag{23}$$

The following transformation is applied, according to the principle of virtual work, to convert the first-order gravitational torques into generalized forces

$$Q_{j_g} = \hat{n}_{q_j} \cdot N_g \tag{24}$$

We will now consider also the effect of the difference in the gravitational forces acting on the two bodies. To a first-order approximation, the gravitational force on body 1, F_{B_g}, is given by Ref. 8 as

$$F_{B_g} = -\frac{\mathrm{GM_e} m_1}{R^2}\left[\left(1 + \frac{r_B \cdot \hat{a}_1}{R}\right)\hat{a}_1 - \frac{r_B}{R}\right] \tag{25}$$

Similarly, the gravitational force on body 2 can be written:

$$F_{C_g} = -\frac{\mathrm{GM_e} m_2}{R^2}\left[\left(1 - \frac{r_C \cdot \hat{a}_1}{R}\right)\hat{a}_1 + \frac{r_c}{R}\right] \tag{26}$$

where r_B is the vector from the system center of mass to the center of mass of body 1 and r_C is the vector from the center of mass of body 2 to the system center of mass. Equations (25) and (26) can be used to show that the total gravitational force on the space station-counterweight system is, to first order not a function of the relative position vectors, r_B and r_C[8]. Therefore there will be no first-order effect of these forces on the equations of motion.

The first-order generalized forces due to gravity-gradient effects will now be evaluated. The $\hat{n}_{q_j}$ for the angular coordinates are developed as (viz. Eqs. (1) and (2))

$$\hat{n}_{\beta_1} = \hat{a}_1;\ \hat{n}_{\beta_2} = \cos\beta_1 \hat{a}_2 + \sin\beta_1 \hat{a}_3$$

$$\hat{n}_{\beta_3} = \sin\beta_2 \hat{a}_1 - \cos\beta_2 \sin\beta_1 \hat{a}_2 + \cos\beta_2 \cos\beta_1 \hat{a}_3 \tag{27a}$$

$$\hat{n}_{\gamma_1} = \hat{a}_1;\ \hat{n}_{\gamma_2} = \cos\gamma_1 \hat{a}_2 + \sin\gamma_1 \hat{a}_3$$

$$\hat{n}_{\gamma_3} = \sin\gamma_2 \hat{a}_1 - \cos\gamma_2 \sin\gamma_1 \hat{a}_2 + \cos\gamma_2 \cos\gamma_1 \hat{a}_3 \tag{27b}$$

$$\hat{n}_{\theta_1} = \hat{a}_3;\ \hat{n}_{\theta_2} = \sin\theta_1 \hat{a}_1 - \cos\theta_1 \hat{a}_2 \tag{27c}$$

Thus the generalized forces, $Q_{q_j} = \hat{n}_{q_j} \cdot N$, under the assumption of small amplitude displacements in the out-of-plane coordinates, can be expressed as follows:

$$Q_{\beta_1} = \hat{a}_1 \cdot N = 0$$

$$Q_{\beta_2} \approx T_2 + \beta_1 T_3; \qquad Q_{\beta_3} \approx T_3 \tag{28a}$$

$$Q_{\gamma_1} = 0$$

$$Q_{\gamma_2} \approx T_2 + \gamma_1 T_3; \qquad Q_{\gamma_3} \approx T_3 \tag{28b}$$

$$Q_{\theta_1} = T_3$$

$$Q_{\theta_2} = -\cos\theta_1 T_2 \tag{28c}$$

It should be noted that $Q_{\beta_1} = Q_{\gamma_1} \equiv 0$ since according to Eq. (21), N_g has no $\hat{a}_1$ component. Similarly, Q_{θ_1} and Q_{θ_2} are also exact expressions within the assumptions previously stated.[8]

E. Development of Linear Equations of Motion

Lagrange's general equations, Eq. (17), were expanded and linearized according to the assumptions previously stated and using the approximate relations, Eqs. (28), for the Q_{j_g} to yield the following:

$\chi-$

$$\ddot{\chi}+\rho_1'\ddot{\alpha}_1+\rho_2'\ddot{\alpha}_2+2(\dot{\theta}_n+\Omega)\dot{\delta}+(\rho_1'+\rho_2')(\dot{\theta}_n+\Omega)^2\chi-\rho_1'(\dot{\theta}_n+\Omega)^2\alpha_1-\rho_2'(\dot{\theta}_n+\Omega)^2\alpha_2=T_3' \tag{29}$$

$\delta-$

$$\ddot{\delta}-(1+\rho_1'+\rho_2')(\dot{\theta}_n+\Omega)^2-2(\dot{\theta}_n+\Omega)\dot{\chi}-2\rho_1'(\dot{\theta}_n+\Omega)\dot{\alpha}_1-2\rho_2'(\dot{\theta}_n+\Omega)\dot{\alpha}_2-(\dot{\theta}_n+\Omega)^2\delta+k_1'(\delta-\delta_0)+k_2'\dot{\delta}=0 \tag{30}$$

α_1-

$$(\rho_1'^2+I_{B_3}')\ddot{\alpha}_1+\rho_1'\rho_2'\ddot{\alpha}_2+\rho_1'\ddot{\chi}+k_{B_3}'\dot{\alpha}_1+2\rho_1'(\dot{\theta}_n+\Omega)\dot{\delta}-\rho_1'(\dot{\theta}_n+\Omega)^2\chi+\rho_1'(1+\rho_2')(\dot{\theta}_n+\Omega)^2\alpha_1+c_{B_3}'\alpha_1-\rho_1'\rho_2'(\dot{\theta}_n+\Omega)^2\alpha_2=T_3' \tag{31}$$

α_2-

$$(\rho_2'^2+I_{C_3}')\ddot{\alpha}_2+\rho_1'\rho_2'\ddot{\alpha}_1+\rho_2'\ddot{\chi}+k_{C_3}'\dot{\alpha}_2+2\rho_2'(\dot{\theta}_n+\Omega)\dot{\delta}-\rho_2'(\dot{\theta}_n+\Omega)^2\chi+\rho_2'(1+\rho_1')(\dot{\theta}_n+\Omega)^2\alpha_2+c_{C_3}'\alpha_2-\rho_1'\rho_1'(\dot{\theta}_n+\Omega)^2\alpha_1=T_3' \tag{32}$$

β_2-

$$\tfrac{1}{2}(I_{B_2}'+I_{B_1}')\ddot{\beta}_2+[(I_{B_2}'+I_{B_1}')\Omega-I_{B_3}'(\dot{\theta}_n+\Omega)]\dot{\beta}_1+k_{B_2}'\dot{\beta}_2+[c_{B_2}'+I_{B_3}'\dot{\theta}_n\Omega-\tfrac{1}{2}(I_{B_1}'-2I_{B_3}'+I_{B_2}')\Omega^2]\beta_2=T_2'+\beta_1T_3' \tag{33}$$

β_1-

$$\tfrac{1}{2}(I_{B_2}'+I_{B_1}')\ddot{\beta}_1+[I_{B_3}'(\dot{\theta}_n+\Omega)-(I_{B_2}'+I_{B_1}')\Omega]\dot{\beta}_2+k_{B_1}'\dot{\beta}_1+[c_{B_1}'+I_{B_3}'\dot{\theta}_n\Omega-\tfrac{1}{2}(I_{B_1}'-2I_{B_3}'+I_{B_2}')\Omega^2]\beta_1=0 \tag{34}$$

θ_2-

$$\ddot{\theta}_2+(1+\rho_1'+\rho_2')(\dot{\theta}_n+\Omega)^2\theta_2=-T_2'\cos\theta_1 \tag{35}$$

γ_2-

$$\tfrac{1}{2}(I_{C_2}'+I_{C_1}')\ddot{\gamma}_2+[(I_{C_2}'+I_{C_1}')\Omega-I_{C_3}'(\dot{\theta}_n+\Omega)]\dot{\gamma}_1+k_{C_2}'\dot{\gamma}_2+[c_{C_2}'+I_{C_3}'\dot{\theta}_n\Omega-\tfrac{1}{2}(I_{C_1}'-2I_{C_3}'+I_{C_2}')\Omega^2]]\gamma_2=T_2'+\gamma_1T_3' \tag{36}$$

γ_1-

$$\tfrac{1}{2}(I_{C_2}'+I_{C_1}')\ddot{\gamma}_1+[I_{C_3}'(\dot{\theta}_n+\Omega)-(I_{C_2}'+I_{C_1}')\Omega]\dot{\gamma}_2+k_{C_1}'\dot{\gamma}_1+[c_{C_1}'+I_{C_3}'\dot{\theta}_n\Omega-\tfrac{1}{2}(I_{C_1}'-2I_{C_3}'+I_{C_2}')\Omega^2]\gamma_1=0 \tag{37}$$

For dimensional consistency in Eqs. (29-37), we note that $T_i'=T_i/\mu\ell_e^2$.

It can be seen by examining Eqs. (22) and (23) that the first-order linear equations with gravity-gradient effects (Eqs. (29-37) now involve periodic coefficients with frequency at twice the spin rate. Also, each of the in-plane equations, Eqs. (29), (31), (32), except the $\ell(\delta)$ equation, now contain forcing terms of constant amplitude on the right side with frequency $2\dot{\theta}_n$. It is also apparent that gravity-gradient effects become more pronounced for small R and are increased as either

$$|I_{B_1}-I_{B_2}| \text{ or } |I_{C_1}-I_{C_2}|$$

or both are increased.

A rigorous stability analysis of a linear system with periodic coefficients could be made using the Floquet theory.[9] For a multi-degree-of-freedom system, the application of Floquet analysis would necessitate extensive computer simulation to examine the moduli of the eigenvalues associated with an augmented matrix and evaluated over a wide range of system parameters. Although the Floquet theory was not applied to this study, the effect of gravity-gradient torques on the system was considered numerically for selected steady-state responses as well as transient responses. Resonance due to gravity-gradient effects is shown to exist for certain special cases, easily identified, and these results are presented in Section IV.

When the gravity-gradient effects are omitted, the linearized out-of-plane equations uncouple completely from the four linearized in-plane equations. At equilibrium for the torque-free system the tensile force in the elastic cable is

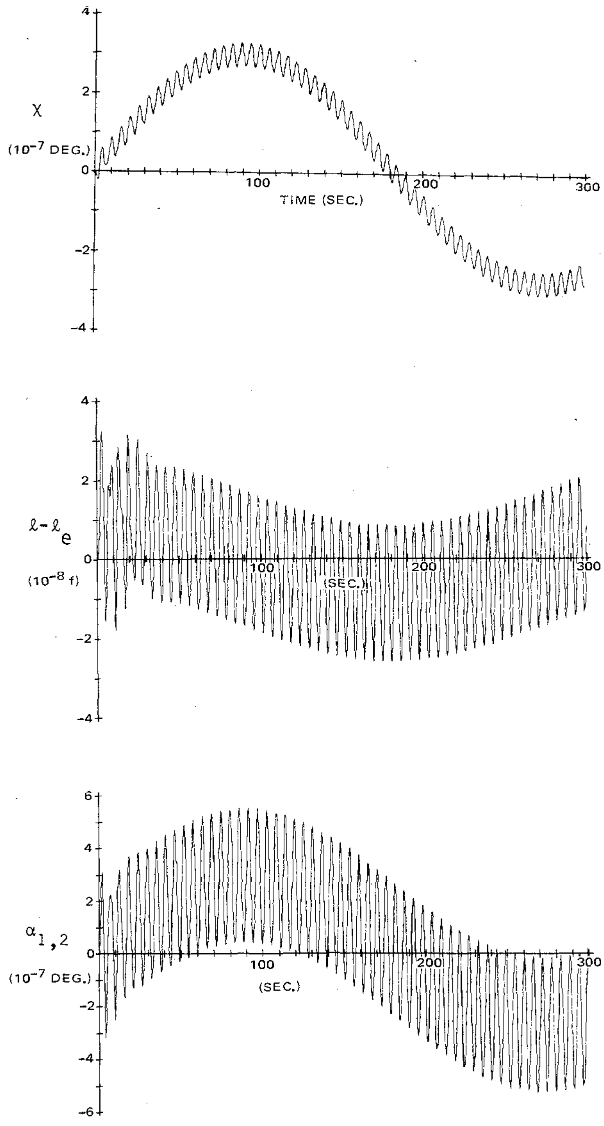

Fig. 3 The in-plane steady-state response due to gravity-gradient effects for the identical system.

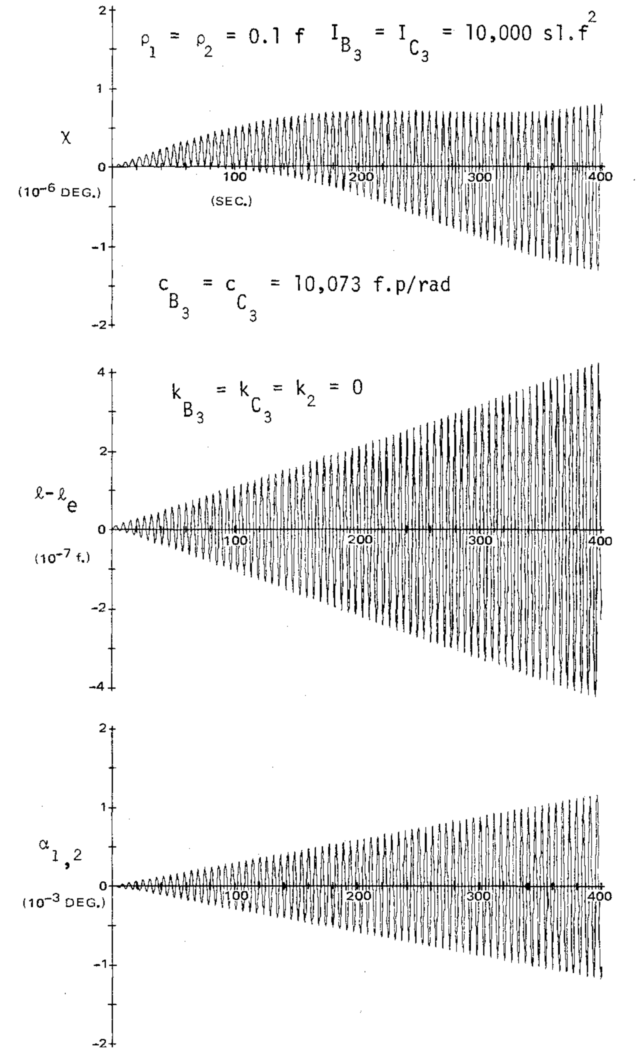

Fig. 4 **Resonant steady-state response for weakly coupled in-plane motion in the presence of gravity-gradient torques.**

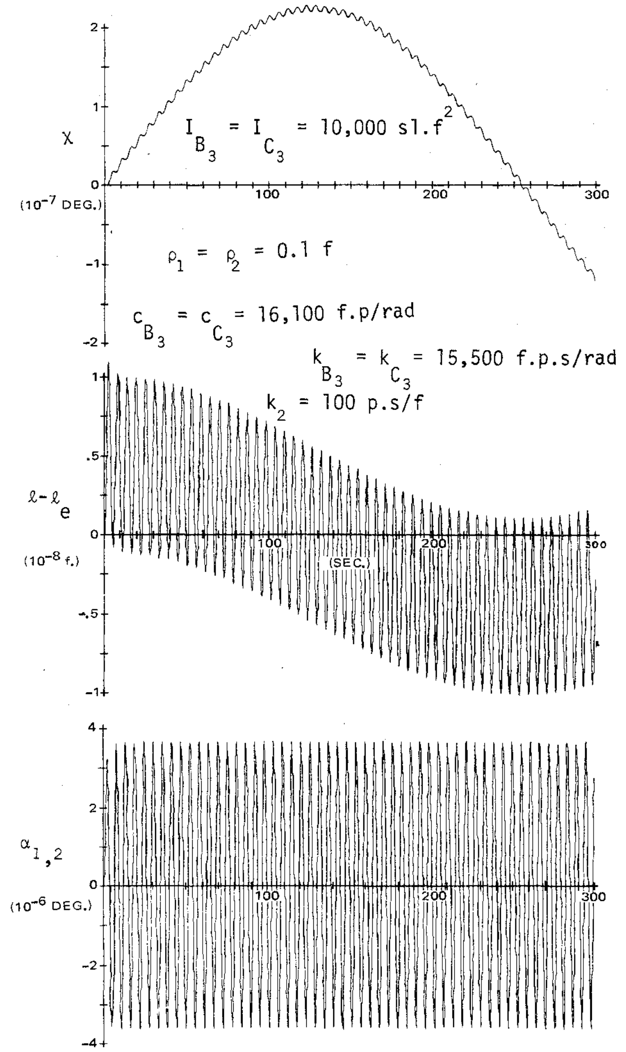

Fig. 5 **Damped resonant steady-state response for weakly coupled in-plane motion with gravity-gradient effects.**

balanced by the centrifugal force associated with the system rotation.[5] Furthermore it is observed that the equation corresponding to the coordinate which describes the displacement of the cable line from the original plane of rotation completely uncouples from all the other equations and indicates simple harmonic motion. This can be interpreted to mean that, for small perturbations on the cable's initial orientation out of the nominal plane of rotation, the system will tend to rotate in a plane inclined to the original plane of rotation without affecting the spin rate or equilibrium cable length.[5]

Stability criteria for the torque free system have been developed analytically and indicate a lower bounds on the value of the cable restoring constant as well as the placement of damping within the system. A complete stability analysis for the torque free system in addition to a parametric optimization of the damping system is presented in Refs. 5 and 7.

IV. Evaluation of Gravity-Gradient Effects

A. Nonresonant Steady-State and Transient Response

The equations of motion incorporating first-order gravity-gradient effects were programed for computer simulation using the IBM 1130 system and employing a fourth-order Runge-Kutta numerical integration subroutine. For the computational step size chosen ($\Delta t = 0.25$ simulated problem seconds) each 1 sec of simulated response required about 30 sec of computer time.

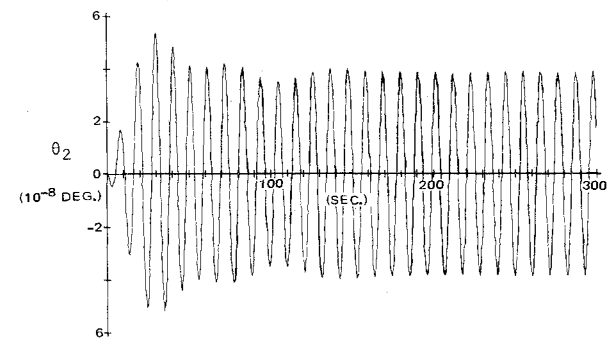

Fig. 6 **Transient response of θ_2 for identical system with gravity-gradient effects.**

The steady-state response (zero initial conditions) was examined for various cases. In all computer runs,

$R = 3.3057 \times 10^7$ ft (2000 nautical miles altitude)

$G \cdot M_e = 1.407528 \times 10^{16}$ ft^3/sec^2

and

$$I_{B_1} - I_{b_2} = 1000 \text{ slug-ft}^2 = I_{C_1} - I_{C_2}$$

Figure 3 shows the steady-state motion of the in-plane variables for the identical end-mass system described in Ref. 5

(Fig. 7). There is no out-of-plane motion for the case of zero initial conditions. The amplitudes of the χ, α_1, and α_2 motions are in the order of 10^{-7}° and for this very small amplitude motion, the response of α_1 is equal to the response of α_2. This can be seen from an examination of Eqs. (31) and (32) which verifies that for the identical system and zero initial conditions, the α_1 and α_2 responses would be expected to remain in phase. In considering the transient response of this system for small intial perturbations (Fig. 7 of Ref. 5), it can be seen that the effect of gravitational torques here would be negligible.

B. Gravity-Gradient Induced Resonance

In all previous cases considered here and in Ref. 5, the following parameters have been assumed

$$\rho_1 = \rho_2 = 12 \text{ ft}, \qquad k_2 = 56.7 \text{ lb/ft}^2$$

$$k_{B_3} = k_{C_3} = 15{,}500 \text{ ft-lb-sec/rad}$$

If $\rho_i/\ell_e \ll 1$, and $k_2 = k_{B_3} = k_{C_3} = 0$, the in-plane motion is only very lightly coupled and completely undamped. For this system, the α_1, α_2 equations, Eqs. (31) and (32), can be approximated by an undamped, uncoupled, forced harmonic oscillator. Conditions resulting in parametric resonance can be readily calculated in closed form. Resonance with respect to the $2\dot{\theta}_n$ forcing frequency was predicted to occur for $I_{B_3} = 10{,}000$ slug-ft^2, $c_{B_3} = c_{C_3} = 10{,}073$ ft-lb/rad, and $\rho_1 = \rho_2 = 0.1$ ft $(\rho_i/\ell_e \ll 1)$. The in-plane steady-state motion for this case is shown in Fig. 4. After 400 sec the χ and cable amplitudes are approximately ten times larger than the amplitudes for those coordinates shown in the steady-state response of the identical system (Fig. 3). At the same time, the amplitudes of the $\alpha_{1,2}$ responses are about four orders of magnitude larger than those for the corresponding identical system in a nonresonant situation (Fig. 3). The amplitudes of all the in-plane coordinates appear to be increasing with time in a secular manner. However, because of the enormous computational times involved here the responses were not simulated beyond 400 sec.

The effect of damping in a resonant situation is shown in Fig. 5. For this case, resonance is predicted when

$$k_2 = 100 \text{ lb-sec/ft}$$

$$k_{B_3} = k_{C_3} = 15{,}500 \text{ ft-lb-sec/rad}$$

$$I_{B_3} = I_{C_3} = 10{,}000 \text{ slug-ft}^2$$

$$c_{B_3} = c_{C_3} = 16{,}100 \text{ ft-lb/rad}$$

A comparison between these results and those of Fig. 4 shows that in the damped response the immediate secular increase of the in-plane amplitudes of the variational coordinates is not apparent, at least within the 300 sec simulated. Furthermore, at the end of 300 sec the amplitude of the $\alpha_{1,2}$ motion is at least three orders of magnitude less than that shown in Fig. 4.

In order to induce out-of-plane motion of the system with gravity-gradient effects, it is necessary to have nonzero initial conditions. With the same initial conditions used for the identical system, i.e., zero initial velocities and

$$\chi = 0 \qquad \alpha_2 = -0.1 \text{ rad}$$

$$\ell - \ell_e = 0.48 \text{ ft} \qquad \beta_1 = 0.1 \text{ rad}$$

$$\alpha_1 = 0.1 \text{ rad} \qquad \beta_2 = 0$$

the amplitudes of transient response in all coordinates (Fig. 7 of Ref. 5) were so much greater than that of the steady-state response for both the in-plane and out-of-plane motion that the effects of gravity-gradient torques were not apparent. However, because the θ_2 coordinate had no initial condition and its motion is less coupled to the other equations in the presence of gravity-gradient effects, the θ_2 response due to gravitational torques for this particular case is shown in Fig. 6. It can be seen that the magnitude of θ_2 during the first 300 sec response is an order of magnitude less than the amplitudes of the angular coordinates shown in Fig. 3.

An examination of system parameters which result in near resonance has been considered numerically[7] by varying the parameters so that the roots of the in-plane system characteristic equation will yield a modal frequency approximately equal to $2\dot{\theta}_n$.

V. Conclusions

The steady-state motion due to first order gravity-gradient effects is shown to be small and its influence on the transient response negligible under nominal nonresonant conditions. Resonance was shown to occur for certain choices of system parameters. For cable attachment lengths which are small in comparison with the cable equilibrium length, the linear equations are less coupled and the effects of resonance can be easily identified. Also, damping may reduce the order of magnitude of the steady-state amplitudes in a resonant situation.

A further examination of gravity-gradient effects could include a redevelopment of the complete gravity-gradient potential energy for this space station-counterwieght system and a more general stability analysis involving Floquet theory.

References

[1] Chobotov, V., "Gravity-Gradient Excitation of a Rotating Cable-Counterweight Space Station in Orbit," *Transactions of the ASME, Ser. E, Journal of Applied Mechanics*, Dec. 1963, pp. 547-554.

[2] Stabekis, P. and Bainum, P. M., "Motion and Stability of a Rotating Space Station-Cable-Counterweight Configuration," *Journal of Spacecraft and Rockets*, Vol. 7, Aug. 1970, pp. 912-918.

[3] Nixon, D. D., "Dynamics of a Spinning Space Station with a Counterweight Connected by Multiple Cables," *Journal of Spacecraft and Rockets*, Vol. 9, Dec. 1972, pp. 896-902.

[4] Anderson, W. W., "On Lateral Cable Oscillations of Cable-Connected Space Stations," NASA TN 5107, Langley Research Center, Hampton, Va., March 1969.

[5] Banium, P. M. and Evans, K. S., "Three-Dimensional Motion and Stability of Two Rotating Cable-Connected Bodies," *Journal of Spacecraft and Rockets*, Vol. 12, April, 1975, pp.242-250.

[6] Robe, T. R. and Kane, T. R., "Dynamics of an Elastic Satellite—I, II, III," *International Journal of Solids & Structures*, 1967, Pergamon Press, Vol. 3, pp. 333-352; 691-703; 1031-1051.

[7] Bainum, P. M. and Evans, K. S., "The Three Dimensional Motion and Stability of a Rotating Space Station-Cable-Counterweight Configuration," Final Report, NASA Grant NGR-09-011-053, Howard University, Dept. of Mechanical Engineering, May 1974.

[8] Robe, T. R., "Stability of Two Tethered Unsymmetrical Earth Pointing Bodies," *AIAA Journal*, Vol. 6, Dec. 1968, pp. 2282-2288.

[9] Meirovitch, Leonard, *Methods of Analytical Dynamics*, McGraw Hill, New York, 1970, pp. 263-272.